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Reconstructing the equation of state for cold nuclear matter from the relationship of any two properties of neutron stars

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A direct method is developed to reconstruct the equation of state for high-density nuclear matter from the relationship between any two properties of neutron stars, such as masses, radii, moments of inertia, baryonic masses, binding energies, gravitational redshifts, and their combinations.

The equation of state (EOS) for supranuclear-density matter with zero temperature is relevant to neutron stars. For supranuclear-density matter, a number of possible scenarios, such as Kaon condensation, pion condensation, hyperonic matter, and strange quark matter, have been proposed. In the near future, lattice QCD is expected to predict the correct EOS for supranuclear-density matter. However, an accelerator-experimental test of the theoretical predictions for the EOS for cold high-density nuclear matter is extremely difficult.

In this light, neutron stars are very unique laboratories. Neutron stars are observed as pulsars, X-ray compact sources, soft γ-ray repeaters, and gravitational wave sources. Binary pulsar observations can determine masses of neutron stars with very high accuracy [1]. From observations of type-I X-ray bursts, kHz quasiperiodic oscillations (QPO's) of low mass X-ray binaries, and so on, some constraints on the radius of neutron stars have been obtained. Actually, the value of the radius itself has been already measured [2]. Very recently, it has been proposed that the radius of neutron stars can be directly measured by the observation of gravitational waves from coalescing neutron star (NS) /black hole (BH) binaries [3,4]. Some combinations of masses and spins of coalescing NS/NS and NS/BH binaries can be determined from gravitational wave observations in the inspiralling phase [5]. Recently, it has been pointed out that the broad peaks at frequencies $\sim 20-40$ Hz of kHz QPO's can be interpreted as the precession frequency due to the Lense-Thirring effect, the relativistic frame-dragging effect due to the spin angular momentum [6]. This suggests the possibility of direct detection of the ratio of the moment of inertia to the mass of neutron stars. Observations of absorption lines in X-ray spectra may reveal the gravitational redshift [7,8]. In some situations, from knowledge of progenitors of supernovae, we may infer the baryonic mass of the remnant neutron stars. In next decade, we will have a good amount of data on the properties of neutron stars.

The structure of neutron stars has been investigated for a large set of EOS's. Arnett and Bowers [9] examined systematically the mass-radius relation of neutron stars for various EOS's. Recently, Lattimer and Prakash [10] examined systematically the relation of masses, radii, moments of inertia, and binding energies for a large number of modern EOS's. An alternative approach is a direct reconstruction of the EOS from the observed properties of neutron stars. Lindblom [11] developed a method to reconstruct the EOS for nuclear matter from neutron-star masses and radii. In this report, we give a direct method to reconstruct the EOS from the relationship of any two properties of neutron stars. We adopt geometrized units such that $c=G=1$.

Before the method of reconstruction is derived, it is helpful to briefly review how observable quantities of neutron stars are obtained for a given EOS. For simplicity, we assume that neutron stars are static and spherically symmetric with no magnetic field. The line element is given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The structure of neutron stars is determined by the Tolman-Oppenheimer-Volkoff (TOV) equations given by

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$$P' = -(P+\epsilon)\frac{m+4\pi r^3 P}{r(r-2m)}, \tag{2}$$

$$m' = 4\pi\epsilon r^2 \tag{3}$$

$$\nu' = -\frac{2}{P+\epsilon}P', \tag{4}$$

where P and ϵ are the pressure and energy density of the matter, m is defined as

$$1 - \frac{2m}{r} = e^{-\lambda}, \tag{5}$$

and the prime denotes the derivative with respect to r. For a given EOS, it is easy to integrate the TOV equations with the initial values $P(0) = P_c$ and $m(0) = 0$. The radius R is the value of r such that $P = 0$ at $r = R$. The mass M is obtained by $M = m(R)$. The moment of inertia I is calculated as [12]

$$I = \frac{1}{6}R^4\frac{\omega'(R)}{\omega(R)}, \tag{6}$$

where $\omega(r)$ is obtained by integrating the following ordinary differential equation

$$\frac{1}{r^4}(r^4 e^{-\frac{\lambda+\nu}{2}}\omega')' + \frac{4}{r}(e^{-\frac{\lambda+\nu}{2}})'\omega = 0, \tag{7}$$

with the initial value $\omega(0) \neq 0$ and $\omega'(0) = 0$. The baryonic mass $m_B N$ and binding energy B are obtained by

$$m_B N = \int_0^R dr e^{\frac{\lambda}{2}} 4\pi\rho r^2, \tag{8}$$

$$B = m_B N - M, \tag{9}$$

where ρ is the rest-mass density that is written as $\rho = m_B n$ using the baryon mass m_B and the baryon number density n. The gravitational redshift z at the surface is given by

$$z = \left(1 - \frac{2M}{R}\right)^{-1/2} - 1 \tag{10}$$

Then, we describe how to reconstruct the EOS from a complete observed data set. The problem is to determine the EOS $\rho = \rho(P)$ for the domain $P_0 \le P \le P_U$, where P_U is some cutoff pressure. We assume that we have a complete relationship $f(Q^{(1)}, Q^{(2)}) = 0$ between observables $Q^{(1)}$ and $Q^{(2)}$ and that we have the correct EOS for $P \le P_0$. We also assume that ρ is a nondecreasing function of P [†]. We prepare N grid points in the domain $[P_0, P_N]$ as $P_0 < P_1 < \cdots < P_i < P_{i+1} < \cdots < P_N = P_U$. Here, as a simple example, we take the grid points as

$$\log P_i = \log P_0 + \frac{i}{N}(\log P_N - \log P_0). \tag{11}$$

Suppose we have the "pointwise exact" EOS for $P \le P_i$. This implies that if the TOV equations are integrated with the initial values $P(0) = P_k$ and $m(0) = 0$ then the obtained $Q_k^{(1)}$ and $Q_k^{(2)}$ satisfy $f(Q_k^{(1)}, Q_k^{(2)}) = 0$ for $0 \le k \le i$. Using some unknown number $\rho_{i+1} = \rho(P_{i+1})$, we interpolate the EOS in the interval $[P_i, P_{i+1}]$. If it is taken into account that the EOS for nuclear matter is locally close to polytropic, then, the best interpolation is the logarithmically linear interpolation

$$\log P = \log P_i + \Gamma_i(\log\rho - \log\rho_i), \tag{12}$$

where Γ_i is the "local polytropic index" given by

$$\Gamma_i \equiv \frac{\log P_{i+1} - \log P_i}{\log\rho_{i+1} - \log\rho_i}. \tag{13}$$

[†] Otherwise, the fluid is locally unstable.

Using this interpolation, ϵ is obtained as functions of $P \in [P_i, P_{i+1}]$ as

$$\epsilon = \rho(1+e), \tag{14}$$

where

$$\rho = \left(\frac{P}{K_i}\right)^{\Gamma_i^{-1}}, \tag{15}$$

$$e = \frac{K_i}{\Gamma_i - 1}\left[\left(\frac{P}{K_i}\right)^{1-\Gamma_i^{-1}} - \rho_i^{\Gamma_i - 1}\right] + e_i, \tag{16}$$

$$K_i \equiv \frac{P_i}{\rho_i^{\Gamma_i}}, \tag{17}$$

e is the specific internal energy and Eq. (16) is obtained from the first law of thermodynamics. Now that we have the EOS for $P \le P_{i+1}$ parametrized by one parameter ρ_{i+1}, we construct the neutron star by integrating the TOV equations with the initial values $P(0) = P_{i+1}$ and $m(0) = 0$. Since the obtained $Q^{(1)}_{i+1}$ and $Q^{(2)}_{i+1}$ are functions of ρ_{i+1}, we denote them as $Q^{(1)}_{i+1}(\rho_{i+1})$ and $Q^{(2)}_{i+1}(\rho_{i+1})$. It is easy to numerically find the root ρ_{i+1} ($\ge \rho_i$) which gives $f(Q^{(1)}_{i+1}(\rho_{i+1}), Q^{(2)}_{i+1}(\rho_{i+1})) = 0$, by the bisection method or Newton-Raphson method with the aid of numerically estimated derivative $df(Q^{(1)}_{i+1}(\rho_{i+1}), Q^{(2)}_{i+1}(\rho_{i+1}))/d\rho_{i+1}$. It is noted that $(\rho_{i+1} - \rho_i)$ may not always be small even for small $(P_{i+1} - P_i)$ because of the possible first order phase transitions. Thus, we find e_{i+1} as

$$e_{i+1} = \frac{K_i}{\Gamma_i - 1}(\rho_{i+1}^{\Gamma_i - 1} - \rho_i^{\Gamma_i - 1}) + e_i. \tag{18}$$

Then we obtain the pointwise exact EOS for $P \le P_{i+1}$.

Repeating these processes, we reconstruct the EOS for $P \le P_N = P_U$. The reconstructed EOS is pointwise exact in the sense that the relationship $f(Q^{(1)}, Q^{(2)}) = 0$ is exactly satisfied by all neutron stars with the central pressure $P \in \{P_0, P_1, \cdots P_N\}$. By taking the limit $N \to \infty$, the reconstructed EOS becomes exact and in this limit the reconstruction does not depend on the interpolation method.

In practice, it is very important how fast this scheme converges as N is increased. It is clear that each ith step reproduces the correct answer if the EOS in the interval $[P_i, P_{i+1}]$ is a polytropic one. This implies that our extrapolation is linear in each interval with respect to $\log P$ and $\log \rho$. For an estimate, we assume that $\log \rho$ is Taylor expandable with respect to $\log P$ as

$$\log \rho = \log \rho_i + \left(\frac{d \log \rho}{d \log P}\right)_i (\log P - \log P_i) + \frac{1}{2}\left(\frac{d^2 \log \rho}{(d \log P)^2}\right)_i (\log P - \log P_i)^2 + \cdots. \tag{19}$$

The error produced in the interval $[P_i, P_{i+1}]$ is estimated for sufficiently large N as

$$(error)_i \simeq \left(\frac{d^2 \log \rho}{(d \log P)^2}\right)_i (\log P_{i+1} - \log P_i)^2. \tag{20}$$

Then the totally accumulated error is bounded from above as

$$\left|\sum_{i=0}^{N-1} (error)_i\right| \lesssim \frac{1}{N}(\log P_N - \log P_0) \sup_{[P_0, P_N]} \left|\frac{d^2 \log \rho}{(d \log P)^2}\right|. \tag{21}$$

Therefore the convergence is as fast as $1/N$.

Next, we describe the reconstruction from a finite data set. In this case, we have only a set of N reliable data pairs

$$\{(Q^{(1)}_1, Q^{(2)}_1), \cdots, (Q^{(1)}_i, Q^{(2)}_i), \cdots, (Q^{(1)}_N, Q^{(2)}_N)\}. \tag{22}$$

The order of data pairs should be determined so that the smaller value of index i corresponds to a neutron star with the lower central pressure. The order of data pairs does not seem to be determined *a priori* except for the fact that M and $m_B N$ are increasing functions of the central pressure for stable neutron stars. Unlike the previous case, we have to determine two unknown numbers P_{i+1} and ρ_{i+1} such that

$$Q^{(1)}_{i+1}(P_{i+1},\rho_{i+1}) = Q^{(1)}_{i+1}, \tag{23}$$

$$Q^{(2)}_{i+1}(P_{i+1},\rho_{i+1}) = Q^{(2)}_{i+1}, \tag{24}$$

where $Q^{(1)}_{i+1}(P_{i+1},\rho_{i+1})$ and $Q^{(2)}_{i+1}(P_{i+1},\rho_{i+1})$ are the values of $Q^{(1)}$ and $Q^{(2)}$, respectively, which are obtained by integrating the TOV equations with $P(0) = P_{i+1}$ and $m(0) = 0$ with the EOS interpolated by Eq. (13)-(17) using the unknown numbers P_{i+1} and ρ_{i+1}. This can be done numerically by the two-dimensional Newton-Raphson method with the aid of the numerically evaluated Jacobian

$$\frac{\partial(Q^{(1)}_{i+1}(P_{i+1},\rho_{i+1}), Q^{(2)}_{i+1}(P_{i+1},\rho_{i+1}))}{\partial(P_{i+1},\rho_{i+1})}. \tag{25}$$

Then we obtain the pointwise exact EOS for $P \le P_{i+1}$. If a set of $\log P_i$ has roughly homogeneous distribution, the convergence is as fast as $1/N$ as we have already seen.

Here, we briefly discuss desired observational accuracy in order to draw some useful conclusions in discriminating between the possible EOS's, based on the figures given by Lattimer and Prakash [10]. The data pairs on masses and radii of neutron stars will be very powerful if the measurement is as accurate as $\Delta M \lesssim 0.1 M_\odot$ and $\Delta R \lesssim 1$ km. For the pairs I/MR^2 and M/R, $\Delta(I/MR^2) \lesssim 0.01$ and $\Delta(M/R) \lesssim 0.01$ would be desired. For the pairs B/M and M/R, $\Delta(B/M) \lesssim 0.01$ and $\Delta(M/R) \lesssim 0.01$ would be desired. However, the above estimate is quite naive, and more systematic and elaborate study is needed. The present accuracy for mass measurement from a radio binary pulsar is $\Delta M \simeq 0.0007 M_\odot$ [1], while that from an X-ray binary is $\Delta M \simeq 0.4 M_\odot$ [13]. For the measurement of radius, $\Delta R \simeq 5.2$ km is obtained if the interstellar absorption of X-ray emission and the distance to the object are fixed [2]. More detailed X-ray spectral analyses and/or the space interferometric mission will dramatically improve the accuracy in determining the radii of neutron stars. Although moments of inertia of neutron stars have not been determined directly, the determination of them will be a complementary and potentially powerful technique.

Finally, we compare the present method of reconstruction from a complete data set with that of Lindblom [11]'s one. (1) The former applies to any two properties of neutron stars while the application of the latter is restricted to the mass-radius relation. (2) The former needs iteration to determine ρ_{i+1} for each ith step, while P_{i+1} and ϵ_{i+1} are determined explicitly and no iteration is needed in the latter. (3) In the former, the relation $f(Q^{(1)}, Q^{(2)}) = 0$ is satisfied at any grid point $P \in \{P_0, P_1, \cdots, P_N\}$ for finite N, while it is not in the latter. (4) In the former, the obtained EOS is of the form $P = P(\rho)$, which is conventional in nuclear physics, while in the latter it is of the form $P = P(\epsilon)$.

In summary, a complete data set of two properties of neutron stars is sufficient to reconstruct the EOS for cold nuclear matter. We have proposed a direct method to reconstruct the EOS. It is crucial to measure more than one property of a neutron star in order to put stringent constraints on the EOS.

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